Engineering Notes

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Suboptimal Linear Filtering

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Introduction

WITH the introduction of modern control theory and various linear filtering techniques, it became apparent that, in practical applications, all of the knowledge of the system could not be included in the filter description. Total system mechanization usually resulted in a model which was too large to be incorporated in a real time data processor. In addition to the size problem, there are other uncertainties which cause the filter to be less than optimal.

The method of filter design, then, requires more than simply describing the plant equations, the observation equations, and their associated statistics. A technique has to be used, in addition to common sense, to evaluate suboptimal vs optimal filter performance.

Early analysis tools made extensive use of Monte Carlo techniques for system performance evaluation. However, with the increase in system complexity Monte Carlo methods became too computationally expensive. As a result, investigators began to look for other methods of suboptimal filter evaluation. The results of some of these studies are presented in Refs. 1–4. Perhaps the most intuitive approaches are those suggested by Nishimura[1] and by Sutherland and Gelb.[2]

These papers discuss the evaluation of a linear filter which is suboptimal, because of errors in the plant and/or measurement statistics or on account of simplifications in the plant and observation matrix definitions. However, the techniques require that all the states needed to model the plant be included in the filter. This results in the evaluation of a $2n \times 2n$ covariance matrix. Some work has been done in the area of filters with reduced state dimensions. However, these techniques require a lot of null arithmetic or special calculations to combine filter and plant statistics in order to evaluate the suboptimal filter's performance.

This Note presents a method of filter evaluation which requires the computation of an $(n+m) \times (n+m)$ covariance matrix where m is the dimension of the state vector which is modeled in the filter. The actual estimation errors come directly from the filter covariance matrices.

The suboptimal filter evaluation algorithm presented here also includes an analysis technique for those cases in which instantaneous reset control is applied to the plant during the estimation process.

Linear Filter Development

Linear filtering, and specifically Kalman filtering, is a well known and understood data smoothing and prediction technique. This short section is included to describe this technique and the notation which will be used in this Note. It is by no means meant to be a complete analytical development of the subject.

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The plant, whose state is to be estimated, satisfies exactly the following matrix differential equation:

$$\dot{\mathbf{x}}(t) = F(t)\mathbf{x}(t) + G(t)\mathbf{w}(t) \tag{1}$$

It is also assumed that the initial states, $\mathbf{x}(t_o)$, and the random process, $\mathbf{w}(t)$, have the following zero-mean gaussian statistics associated with them:

$$E[\mathbf{x}(t_o)\mathbf{x}^T(t_o)] = P(t_o) = P_o \tag{2}$$

$$E[\mathbf{w}(t)\mathbf{w}^T(\tau)] = Q(t)\delta(t-\tau) \tag{3}$$

$$E[\mathbf{x}(t)\mathbf{w}^T(t)] = 0 \tag{4}$$

The measurement process, which will be used to estimate the states, is described by the following equation:

$$\mathbf{z}(t_i) = H(t_i)\mathbf{x}(t_i) + \mathbf{v}(t_i) \tag{5}$$

The statistics associated with the observations are

$$E[\mathbf{v}(t_i)] = 0 \tag{6}$$

$$E[\mathbf{v}(t_i)\mathbf{v}^T(t_j)] = R(t_i)\delta_{ij} \tag{7}$$

$$E[\mathbf{x}(t_i)\mathbf{v}^T(t_i)] = 0 \tag{8}$$

The equations which constitute the filter algorithm may be divided into two distinct operations. The first operation is associated with propagating the state estimate and associated statistics between updates. Equations (9) and (10) are the differential equations used to accomplish this operation for the discrete Kalman filter.

$$\dot{\hat{\mathbf{x}}}(t) = F(t)\hat{\mathbf{x}}(t) \tag{9}$$

$$\dot{P}(t) = F(t)P(t) + P(t)F^T(t) + G(t)Q(t)G(t)^T \tag{10}$$

where

$$P(t) = E[(\mathbf{x}(t) - \hat{\mathbf{x}}(t))(\mathbf{x}(t) - \hat{\mathbf{x}}(t))^T] \tag{11}$$

The initial conditions associated with $\hat{\mathbf{x}}$ and P at the start of the propagation interval which spans the time t_i to t_{i+1} are

$$\hat{\mathbf{x}}(t_i) = \hat{\mathbf{x}}(t_{i/i})\dagger \tag{12}$$

$$P(t_i) = P(t_{i/i}) \tag{13}$$

At time t_{i+1} the filter must perform the second operation which is to incorporate the observation, $\mathbf{z}(t_{i+1})$, with the state estimate $\hat{\mathbf{x}}(t_{i+1}) = \hat{\mathbf{x}}(t_{i+1/i})$. This is accomplished using the following equations:

$$\hat{\mathbf{x}}(t_{i+1/i+1}) = \hat{\mathbf{x}}(t_{i+1/i}) + K(t_{i+1})(\mathbf{z}(t_{i+1}) - H(t_{i+1})\hat{\mathbf{x}}(t_{i+1/i})) \tag{14}$$

$$P(t_{i+1/i+1}) = (I - K(t_{i+1})H(t_{i+1}))P(t_{i+1/i}) \times (I - K(t_{i+1})H(t_{i+1}))^T + K(t_{i+1})R(t_{i+1})K(t_{i+1})^T \tag{15}$$

The state estimate and covariance matrix resulting from Eqs. (14) and (15) are used as initial conditions to Eqs. (9) and (10) and the process repeated. The filter gain, $K(t_{i+1})$, is computed based on the covariance matrix $P(t_{i+1/i})$ as well as $H(t_{i+1})$ and $R(t_{i+1})$.

Partitioning the System State Vector

The equations of the previous section assumed that the entire process description, Eq. (1) and observation matrix description, Eq. (5), can be mechanized in the data filter. The purpose of this section is to introduce the concept of a partitioned state vector and the notation to be used in the following paragraphs. This vector is made up of those elements, $\mathbf{x}_1$, which will be included in the suboptimal filter and the vector $\mathbf{x}_2$, which contains the unmodeled states. That is

† The vector $\hat{\mathbf{x}}(t_{i/j})$ is the best estimate of $\mathbf{x}$ at time t_i given j measurements.

$$\mathbf{x} = \begin{bmatrix} \mathbf{x}_1 \\ \cdots \\ \mathbf{x}_2 \end{bmatrix} \tag{16}$$

The state vectors $\mathbf{x}_1$ and $\mathbf{x}_2$ satisfy the following differential equations:

$$\begin{bmatrix} \dot{\mathbf{x}}_1 \\ \cdots \\ \dot{\mathbf{x}}_2 \end{bmatrix} = \begin{bmatrix} F_{11} & \vdots & F_{12} \\ \cdots & & \cdots \\ F_{21} & \vdots & F_{22} \end{bmatrix} \begin{bmatrix} \mathbf{x}_1 \\ \cdots \\ \mathbf{x}_2 \end{bmatrix} + \begin{bmatrix} G_1 \\ \cdots \\ G_2 \end{bmatrix} \mathbf{w} \tag{17}$$

The following equation defines the covariance matrices pertaining to the parts of the partitioned state vector:

$$E\left\{ \begin{bmatrix} \mathbf{x}_1(t_o) \\ \cdots \\ \mathbf{x}_2(t_o) \end{bmatrix} [\mathbf{x}_1{}^T(t_o) \vdots \mathbf{x}_2{}^T(t_o)] \right\} = P_o = \begin{bmatrix} U & \vdots & V \\ \cdots & & \cdots \\ V^T & \vdots & W \end{bmatrix} \tag{18}$$

Based on Eqs. (17) and (18), the following equations may be written to describe the time-varying statistics of $\mathbf{x}_1$ and $\mathbf{x}_2$.‡

$$\dot{U} = F_{11}U + UF_{11}{}^T + F_{12}V^T + VF_{12}{}^T + G_1QG_1{}^T \tag{19}$$

$$\dot{V} = F_{11}V + F_{12}W + UF_{21}{}^T + VF_{22}{}^T + G_1QG_2{}^T \tag{20}$$

$$\dot{W} = F_{21}V + V^TF_{21}{}^T + F_{22}W + WF_{22}{}^T + G_2QG_2{}^T \tag{21}$$

The observation vector, $\mathbf{z}$, may also be written in terms of $\mathbf{x}_1$ and $\mathbf{x}_2$. That is

$$\mathbf{z} = H_1\mathbf{x}_1 + H_2\mathbf{x}_2 + \mathbf{v} \tag{22}$$

The statistics associated with $\mathbf{v}$ are given in Eqs. (6–8).

The Suboptimal Filter Equations

The filter which is to be mechanized to estimate $\mathbf{x}$, based on the measurement vector, $\mathbf{z}$, will assume the following plant model:

$$\dot{\mathbf{x}}_1 = F_{11_c}\mathbf{x}_1 + G_{1_c}\mathbf{w} \tag{23}$$

The following statistics will be assumed for the zero mean processes $\mathbf{x}_1$ and $\mathbf{w}$.

$$E[\mathbf{x}_1(t_o)\mathbf{x}_1{}^T(t_o)] = P_c(t_o) \tag{24}$$

$$E[\mathbf{w}(t)\mathbf{w}^T(\tau)] = Q_c(t)\,\delta(t-\tau) \tag{25}$$

$$E[\mathbf{x}_1(t)\mathbf{w}^T(t)] = 0 \tag{26}$$

In addition, the model for the observation vector will be the one defined by Eq. (27)

$$\mathbf{z} = H_{1_c}\mathbf{x}_1 + \mathbf{v} \tag{27}$$

where

$$E[\mathbf{v}(t_i)\mathbf{v}^T(t_j)] = R_c(t_i)\,\delta_{ij} \tag{28}$$

The filter will use a gain K_c which is either predetermined or generated using the Kalman gain formulation of Eq. (29).

$$K_c = P_c(t_{i+1/i})H_{1_c}{}^T(H_{1_c}P_c(t_{i+1/i})H_{1_c}{}^T + R_c)^{-1} \tag{29}$$

The suboptimal filter will use the following sets of equations to process the measurements: between updates

$$\dot{\hat{\mathbf{x}}}_1 = F_{11_c}\hat{\mathbf{x}}_1 \tag{30}$$

$$\dot{P}_c = F_{11_c}P_c + P_cF_{11_c}{}^T + G_{1_c}Q_cG_{1_c}{}^T \tag{31}$$

at an update

$$\hat{\mathbf{x}}_1(t_{i+1/i+1}) = \hat{\mathbf{x}}_1(t_{i+1/i}) + K_c(\mathbf{z} - H_{1_c}\hat{\mathbf{x}}_1(t_{i+1/i})) \tag{32}$$

$$P_c(t_{i+1/i+1}) = (I - K_cH_{1_c})P_c(t_{i+1/i})(I - K_cH_{1_c})^T + K_cR_cK_c{}^T \tag{33}$$

The statistics on the state estimate, P_c, will be incorrect due to the modeling errors which were introduced. That is

$$E[(\mathbf{x}_1 - \hat{\mathbf{x}}_1)(\mathbf{x}_1 - \hat{\mathbf{x}}_1)^T] \neq P_c \tag{34}$$

It is the purpose of the following section to determine the exact statistics associated with the estimation error.

The Actual Estimation Errors

The filter estimate error, after an update, satisfies the following equation:

$$\mathbf{x}_1 - \hat{\mathbf{x}}_1(t_{i+1/i+1}) = \mathbf{x}_1 - \hat{\mathbf{x}}_1(t_{i+1/i}) - K_c(H_1\mathbf{x}_1 + H_2\mathbf{x}_2 + \mathbf{v} - H_{1_c}\hat{\mathbf{x}}_1(t_{i+1/i})) \tag{35}$$

By making the definition that

$$H_1 = H_{1_c} + \delta H_1 \tag{36}$$

Equation (35) may be written as

$$\mathbf{x}_1 - \hat{\mathbf{x}}_1(t_{i+1/i+1}) = (I - K_cH_{1_c})(\mathbf{x}_1 - \hat{\mathbf{x}}_1(t_{i+1/i})) - K_c\delta H_1\mathbf{x}_1 - K_cH_2\mathbf{x}_2 - K_c\mathbf{v} \tag{37}$$

When Eq. (37) is multiplied by its transpose and the expected value of the result taken, the covariance matrix of the actual estimation error after an update may be related to the statistics prior to the update using Eq. (38)

$$\begin{aligned} P(t_{i+1/i+1}) &= (I - K_cH_{1_c})P(t_{i+1/i})(I - K_cH_{1_c})^T + K_cRK_c{}^T + \\ &K_c(\delta H_1U\delta H_1{}^T + \delta H_1VH_2{}^T + H_2V^T\delta H_1{}^T + H_2WH_2{}^T)K_c{}^T - \\ &(I - K_cH_{1_c})(X(t_{i+1/i})\,\delta H_1{}^T + Y(t_{i+1/i})H_2{}^T)K_c{}^T - \\ &K_c(\delta H_1X^T(t_{i+1/i}) + H_2Y^T(t_{i+1/i}))(I - K_cH_{1_c})^T \end{aligned} \tag{38}$$

where

$$E[(\mathbf{x}_1 - \hat{\mathbf{x}}_1)(\mathbf{x} - \hat{\mathbf{x}}_1)^T] \triangleq P \tag{39}$$

$$E[(\mathbf{x}_1(t) - \hat{\mathbf{x}}_1(t))\mathbf{x}_1{}^T(t)] \triangleq X(t) \tag{40}$$

$$E[(\mathbf{x}_1(t) - \hat{\mathbf{x}}_1(t))\mathbf{x}_2{}^T(t)] \triangleq Y(t) \tag{41}$$

The only remaining computations, at an update, are those required to determine $X(t_{i+1/i+1})$ and $Y(t_{i+1/i+1})$. They are found by multiplying Eq. (37) by $\mathbf{x}_1{}^T$ and then by $\mathbf{x}_2{}^T$ and taking the expected values. The results are

$$X(t_{i+1/i+1}) = (I - K_cH_{1_c})X(t_{i+1/i}) - K_c(\delta H_1{}^TU + H_2V^T) \tag{42}$$

$$Y(t_{i+1/i+1}) = (I - K_cH_{1_c})Y(t_{i+1/i}) - K_c(\delta H_1{}^TV + H_2W) \tag{43}$$

The covariance matrices, P, X, and Y are reset at an update using Eqs. (38, 42, and 43). The next set of relationships which is required are the differential equations to be used to propagate P, X, and Y between updates. If Eq. (30) is subtracted from that portion, Eq. (17) which governs $\dot{\mathbf{x}}$, the following equation may be written:

$$\begin{bmatrix} \dot{\mathbf{x}}_1 - \dot{\hat{\mathbf{x}}}_{1_c} \\ \cdots \\ \dot{\mathbf{x}}_1 \\ \cdots \\ \dot{\mathbf{x}}_2 \end{bmatrix} = \begin{bmatrix} F_{11_c} & \delta F_{11} & \vdots & F_{12} \\ \cdots & \cdots & & \cdots \\ 0 & F_{11} & \vdots & F_{12} \\ \cdots & \cdots & & \cdots \\ 0 & F_{21} & \vdots & F_{22} \end{bmatrix} \begin{bmatrix} \mathbf{x}_1 - \hat{\mathbf{x}}_{1_c} \\ \cdots \\ \mathbf{x}_1 \\ \cdots \\ \mathbf{x}_2 \end{bmatrix} + \begin{bmatrix} G_1 \\ \cdots \\ G_1 \\ \cdots \\ G_2 \end{bmatrix} \mathbf{w} \tag{44}$$

where

$$\delta F_{11} = F_{11} - F_{11_c} \tag{45}$$

Equation (44) may now be used as the definition of the time rate of change of entire state vector and the following equations may be used to propagate P, X, and Y:

$$\dot{P} = F_{11_c}P + PF_{11_c}{}^T + \delta F_{11}X^T + X\delta F_{11}{}^T + F_{12}Y^T + YF_{12}{}^T + G_1QG_1{}^T; \quad P(t_o) = U(t_o) \tag{46}$$

$$\dot{X} = F_{11_c}X + \delta F_{11}U + F_{12}V^T + XF_{11}{}^T + YF_{12}{}^T + G_1QG_1{}^T; \quad X(t_o) = U(t_o) \tag{47}$$

$$\dot{Y} = F_{11_c}Y + \delta F_{11}V + F_{12}W + XF_{21}{}^T + YF_{22}{}^T + G_1QG_2{}^T; \quad Y(t_o) = V(t_o) \tag{48}$$

Equations (19–21) define the time rate of change of the matrices U, V, and W.

Plant Control

There are several types of control which may be applied based on the best estimate of $\mathbf{x}_1$, which the filter feels it is producing. Of interest here is the application of an instantaneous control to the plant based on $\hat{\mathbf{x}}_1(t)$. The reset control is usually performed immediately after an update but that restriction will not be applied here. Some or all of the states may be reset using Eq. (49)

$$\mathbf{x}_1{}'(t) = \mathbf{x}_1(t) - \Gamma\hat{\mathbf{x}}_1(t)\S \tag{49}$$

‡ The time notation has been dropped here and in the sequel, to improve readability of the equations. It will be included only where a doubt as to the proper value may exist.

§ The prime (′) notation is used to indicate the value of a vector or matrix immediately after the reset control is applied.

If $\mathbf{x}_1$ is reset using Eq. (49), then a similar type of reset must be applied to $\hat{\mathbf{x}}_1(t)$ at the same time. That is

$$\hat{\mathbf{x}}_1'(t) = \hat{\mathbf{x}}_1(t) - \Gamma\hat{\mathbf{x}}_1(t) \triangleq \Delta\hat{\mathbf{x}}_1(t) \tag{50}$$

In this section the effect of this type of control on the covariance matrices used in the suboptimal filter evaluation will be discussed. This is best done by rewriting Eq. (49) as

$$\mathbf{x}_1'(t) = \Gamma(\mathbf{x}_1(t) - \hat{\mathbf{x}}_1(t)) + \Delta\mathbf{x}_1(t) \tag{51}$$

The definitions of P, X, Y, U, V, and W may be used to derive the relationships between the covariance, before and after the reset. The expression relating X' to the original covariance elements as well as Γ and Δ will be derived here.

Following the definition of X in Eq. (40)

$$X = E[(\mathbf{x}_1 - \hat{\mathbf{x}}_1)x_1^T] \tag{40}$$

it is logical to define X' as

$$X' = E[(\mathbf{x}_1' - \hat{\mathbf{x}}_1')^T \mathbf{x}_1'^T] \tag{52}$$

However, if Eq. (50) is subtracted from Eq. (49) and the result substituted in Eq. (52), the following expression is obtained:

$$X' = E[(\mathbf{x}_1 - \hat{\mathbf{x}}_1)[\Gamma(\mathbf{x}_1 - \hat{\mathbf{x}}_1) + \Delta\mathbf{x}_1]^T] \tag{53}$$

This equation may now be expanded to yield the following relationship between X', P, X, Γ, and Δ:

$$X' = P\Gamma^T + X\Delta^T \tag{54}$$

The similar relationships for P', Y', U', V', and W' are given below.

$$P' = P \tag{55}$$

$$Y' = Y \tag{56}$$

$$U' = \Gamma P \Gamma^T + \Gamma X \Delta^T + \Delta X^T \Gamma^T + \Delta U \Delta^T \tag{57}$$

$$V' = \Gamma Y + \Delta V \tag{58}$$

$$W' = W \tag{59}$$

Equations (54–59) may be used to evaluate the effect of reset control on the performance of a system using a suboptimal filter.

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Thermal Analysis for Antenna Window in AGM Nosetip

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I. Introduction

COMPUTER simulation[1,2] of one trajectory for a current air-to-ground missile design, which includes a phenolic refrasil heat-shield nosetip, indicates that the phenolic resin binder will char and decompose to a slight extent. However, even a very thin carbonaceous char layer is electrically conductive and highly attenuating to electromagnetic signals. Thus, for flight tests with a telemetry antenna in the nose, a noncharring antenna window must be inserted. This Note describes the nosetip redesign for insertion of a noncharring antenna window and discusses the thermal considerations leading to the redesign.

The most severe trajectories for this vehicle are similar to but less severe than the late stages of a typical re-entry vehicle flight. The operational heat shield easily meets the usual goals of: 1) low density; 2) low thermal conductivity; 3) high specific heat; 4) high resin decomposition energy; and 5) high emissivity.

To prevent charring, a slip cast fused silica antenna window designed under the following constraints was inserted in the nosetip: 1) the existing outer body contour must remain the same to preserve total missile flight parameters; 2) enough of the phenolic refrasil must remain under the slip cast fused silica to serve as structural support for underlying antenna parts; and 3) the combined thicknesses of slip cast fused silica and phenolic refrasil must remain below $\frac{3}{4}$ in. to allow sufficient volume for the antenna itself. The final design shown in Fig. 1 includes an intermediate layer of cellular silicone whose necessity will be explained later.

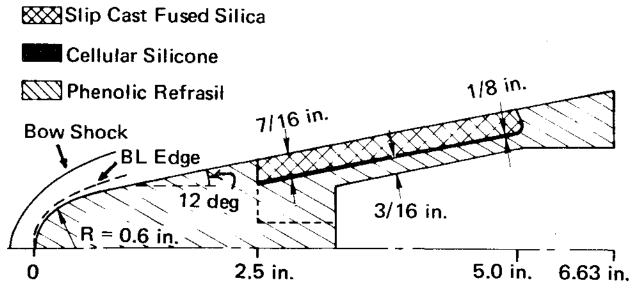

Fig. 1 AGM nosetip heat shield.

II. Aerodynamic Heating

The velocity of the vehicle remains within the low supersonic ($M_\infty \leq 3.5$) regime, and therefore the ionization which occurs in the inviscid flow aft of the bow shock is insignificant. The semiballistic trajectory considered here causes free-flight recovery temperatures on the order of 3000°F with forebody hot wall heating rates of about 25 Btu/ft^2-sec and integrated heat input of about 1000 Btu/ft^2. For the present problem, convective heating dominates to an extent which makes radiative heating from behind the bow shock negligible by comparison.

A code called BLUNTY[1] was selected for its fast running times and reasonably accurate predictions in keeping with knowledge of the input data. Basically BLUNTY functions as follows.

Exact inviscid flowfield solutions including shock shape and static pressure distributions over a wide range of Mach numbers are built into BLUNTY as cubic spline fits and tabular data, respectively. A streamtube mass balance is used to find boundary-layer edge properties which for equilibrium flows at zero angle of attack may be determined by entropy and pressure. The user of BLUNTY may choose between air in thermochemical equilibrium or an ideal gas. Cold wall heating rates are computed in BLUNTY with the Fay and Riddell equation[3] at the stagnation point; the Kemp, Rose, and Detra equation[4], for laminar flow; and a modification of the Rose, Probstein, and Adams equation[5] for turbulent flow.

Though almost a century has elapsed since Osborne Reynolds described boundary-layer transition, it is still the least understood of the important phenomena affecting heat-shield design.[6] Nevertheless, trial calculations indicate that assuming all-turbulent flow coupled with other conservatism leads to a

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